MODULES OVER POLYDISC ALGEBRAS

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Introduction. We shall be concerned with certain analytic covers of the polydisc $U^N=\{(z_1,\ldots,z_N) : |z_1|<1,\ldots,|z_N|<1\}$ in complex N-space, C^N. We will consider a collection $\mathscr{K}_N$ of objects Δ: $\Delta\in\mathscr{K}_N$ if Δ is an open subset of a Stein manifold and if there is a neighborhood Ω of $\bar{\Delta}$ and a proper holomorphic map Φ from Ω onto a neighborhood of $\bar{U}^N$ which satisfies

(1) $\Delta=\Phi^{-1}(U^N)$, and

(2) Φ is a local homeomorphism at each point of $\Phi^{-1}(T^N)$.

Here $T^N=\{(z_1,\ldots,z_N) : |z_1|=\cdots=|z_N|=1\}$, the distinguished boundary of U^N. It follows from the definition of "proper" (compact sets have compact inverse images) that $\bar{\Delta}$ is compact.

Given a $\Delta\in\mathscr{K}_N$ and an associated Φ, the triple $(\Delta, \Phi|\Delta, U^N)$ is an analytic cover, in the sense of [5]. We will make frequent use of the properties of such covers, often without explicit reference. One of their important properties is that Φ has a well-defined multiplicity: there is an integer λ and an analytic variety V in U^N (dim $V<N$) such that each point of $U^N\backslash V$ has exactly λ preimages in Ω.

We will study two algebras naturally associated with Δ,

$$A(\Delta) = \{f\in C(\bar{\Delta}) : f \text{ is holomorphic in } \Delta\}$$

and

$$H^\infty(\Delta) = \{f : f \text{ is bounded and holomorphic in } \Delta\}.$$

These algebras are modules over their subalgebras

$$\Phi^*A(U^N) = \{f\circ\Phi : f\in A(U^N)\}$$

and

$$\Phi^*H^\infty(U^N) = \{f\circ\Phi : f\in H^\infty(U^N)\}.$$

In §I we show that *these modules are actually free, and that their rank equals the multiplicity of* Φ (Theorem I.4). Certain analytic consequences of this are also developed in §I. An example shows that the above result can fail for maps Φ which violate condition (2).

In the one-dimensional case, results of this kind have been obtained by Alling. The elements of $\mathscr{K}_1$ are simply the Riemann surfaces considered in [1].

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Given Δ', $\Delta \in \mathscr{K}_N$ and an associated $\Phi\colon \Delta \to U^N$, we write $\Psi \in \mathscr{M}(\Delta', \Delta)$ if Ψ is a proper holomorphic map of a neighborhood of $\overline{\Delta}'$ onto a neighborhood of $\overline{\Delta}$ which is locally a homeomorphism at each point of $\Psi^{-1}(\Phi^{-1}(T^N))$ and which satisfies $\Delta' = \Psi^{-1}(\Delta)$. The set $\mathscr{M}(\Delta', \Delta)$ depends only on Δ and Δ' and not on Φ, for, as we shall see, the set $\Phi^{-1}(T^N)$ is independent of Φ.

If $\Psi \in \mathscr{M}(\Delta', \Delta)$, then $A(\Delta')$ and $H^\infty(\Delta')$ are again modules over $\Psi^*A(\Delta)$ and $\Psi^*H^\infty(\Delta)$ respectively. This module structure will be studied in a subsequent paper.

§II deals with the case in which both Δ and Δ' are polydiscs. The special form of proper holomorphic maps from U^k to U^N is discussed. Combining this with the results of §I, the following extension theorem is obtained:

THEOREM. *Let Φ be a biholomorphic map of U^k onto a closed analytic submanifold V of U^N. If Φ is holomorphic and one-to-one in a neighborhood of $\overline{U}^k$, then there is a bounded linear operator $E\colon H^\infty(U^k) \to H^\infty(U^N)$ which maps $A(U^k)$ into $A(U^N)$, and which extends functions on V to functions in U^N in the sense that*

$$(Ef) \circ \Phi = f$$

for every $f \in H^\infty(U^k)$.

An example shows that such an extension need not exist if Φ behaves badly near the boundary.

In §III, which is independent of the preceding ones, we obtain a rather general theorem on extending bounded holomorphic functions from an element of $\mathscr{K}_1$ embedded in a U^M to bounded holomorphic functions in U^M. On the one hand, this is a generalization of a previous result [13], and on the other, it is a model of what we would like to prove about extending functions from a $\Delta \in \mathscr{K}_N$ which is embedded in a U^M.

Certain notations will be used consistently. If $\mathfrak{M}$ is a complex manifold, and if $z \in \mathfrak{M}$, then $\mathscr{O}(\mathfrak{M})$, $\mathscr{O}_{\mathfrak{M}}$, and $\mathscr{O}_z$ will denote, respectively, the algebra of all holomorphic functions on $\mathfrak{M}$, the sheaf of germs of such functions, and the stalk of this sheaf at z. A holomorphic map $\Phi\colon \mathfrak{M} \to \mathfrak{N}$, where $\mathfrak{M}$ and $\mathfrak{N}$ are complex manifolds of dimension m and n, is said to be *nonsingular at* $z \in \mathfrak{M}$ if there exist neighborhoods V of z and W of $\Phi(z)$ and biholomorphic maps α and β of V and W onto open subsets of $\boldsymbol{C}^m$ and $\boldsymbol{C}^n$, respectively, such that $\beta \circ \Phi \circ \alpha^{-1}$ is nonsingular in the sense of [5; p. 16]. If Φ is nonsingular at each point of its domain, we say simply that Φ is *nonsingular*.

I. **Analytic covers of polydiscs.**

We begin this section with some general properties of elements of $\mathscr{M}(\Delta', \Delta)$ and then apply these to the special case that Δ is a polydisc.

In the definition of the elements Δ of $\mathscr{K}_N$ we required the existence of a neighborhood Ω of $\overline{\Delta}$ and a proper holomorphic map Φ from Ω to a neighborhood of $\overline{U}^N$ in $\boldsymbol{C}^N$; this Φ is to be a local homeomorphism at each point of $\Phi^{-1}(T^N)$. It is important to observe that the set $\Phi^{-1}(T^N)$ can be described in terms of the algebra $A(\Delta)$; thus $\Phi^{-1}(T^N)$ does not depend on the particular choice of Φ:

I.1. LEMMA $\Phi^{-1}(T^N)$ *is the Shilov boundary for* $A(\Delta)$.

Proof. Let $B=\Phi^{-1}(T^N)$. We claim that every $z\in B$ is a peak point for some element of $A(\Delta)$. If $z\in B$, there exists $f\in A(U^N)$ such that $f(\Phi(z))=1$ but $|f(\zeta)|<1$ at every other point $\zeta\in \overline{U}^N$. Since $A(\Delta)$ separates points on $\overline{\Delta}$, there exists $g\in A(\Delta)$ such that $g(z)=1$ but $g(w)=0$ at every other point $w\in \Phi^{-1}(\Phi(z))$. If V is a neighborhood (in $\overline{\Delta}$) of z, if M is a sufficiently large positive integer, and if $h=(f\circ\Phi)^M g$, then $h(z)=1$, $|h|<9/8$ on $\overline{\Delta}$, and $|h|<1/8$ off V. Thus Bishop's characterization of peak points [2, Theorem 2] shows that z is a peak point for $A(\Delta)$. Hence B is a subset of the Shilov boundary.

To prove that B is a boundary for $A(\Delta)$ we must show that $|f|$ attains its maximum on B if $f\in A(\Delta)$. To do this, suppose $0<t<1$, let T_t^N be the distinguished boundary of the polydisc U_t^N ($z\in U_t^N$ if and only if $t^{-1}z\in U^N$), and put $\Delta_t=\Phi^{-1}(U_t^N)$. If $f\in A(\Delta)$, then $f\in \mathcal{O}(\overline{\Delta}_t)$, so [6] implies that there is $z_t\in\Phi^{-1}(T_t^N)$ such that $|f(z_t)|=\sup\{|f(z)| : z\in\Delta_t\}$. The set $\{z_t : 0<t<1\}$ has a limit point $z_1\in B$, and $|f|$ attains its maximum at z_1. This completes the lemma.

Now suppose $\Delta', \Delta\in\mathcal{K}_N$, $\Phi\in\mathcal{M}(\Delta',\Delta)$, Ω' is a neighborhood of $\overline{\Delta}'$ which Φ maps properly onto a neighborhood Ω of $\overline{\Delta}$, and $\Omega'=\Phi^{-1}(\Omega)$. The mapping Φ gives rise to a sheaf $\mathcal{S}$ on Ω (called the *direct image sheaf* of $\mathcal{O}_{\Omega'}$), in the following manner: if V is an open set in Ω, the sections of $\mathcal{S}$ over V are the holomorphic functions on $\Phi^{-1}(V)$. The set $\Gamma(V,\mathcal{S})=\mathcal{O}(\Phi^{-1}(V))$ of these sections may be regarded as an $\mathcal{O}(V)$-module: If $f\in\mathcal{O}(V)$ and $g\in\Gamma(V,\mathcal{S})$, define fg to be the element $(f\circ\Phi)g$ of $\Gamma(V,\mathcal{S})$.

Since the fibers $\Phi^{-1}(\Phi(z))$ are finite, [11, Theorem 7, p. 81] (which as the referee has pointed out to us, was proved by Oka [16]) shows that $\mathcal{S}$ is a coherent analytic sheaf. For our purposes, it is necessary to know somewhat more:

I.2. LEMMA 2. *If* Φ *has multiplicity* λ, *then* $\mathcal{S}$ *is a locally free sheaf of rank* λ *over* $\mathcal{O}_\Omega$.

Proof. Explicitly, the assertion is that every $z\in\Omega$ has a neighborhood V such that $\mathcal{S}|V$ is isomorphic to the direct sum of λ copies of $\mathcal{O}_V$. Since $\mathcal{S}$ is coherent, it is enough to prove that every stalk $\mathcal{S}_z$ ($z\in\Omega$) is isomorphic (as an $\mathcal{O}_z$-module) to the direct sum of λ copies of $\mathcal{O}_z$.

Indeed, let us assume that this last statement has been proved. For a fixed $z\in\Omega$, let $(s_1)_z,\dots,(s_\lambda)_z\in\mathcal{S}_z$ be germs that constitute a free basis of $\mathcal{S}_z$ over $\mathcal{O}_z$. Since $\mathcal{S}$ is coherent, z has a neighborhood V_1 such that $(s_1)_\zeta,\dots,(s_\lambda)_\zeta$ generate $\mathcal{S}_\zeta$ for every $\zeta\in V_1$. Let $\mathcal{R}$ be the sheaf of relations of the sections $s_1,\dots,s_\lambda\in\Gamma(V_1,\mathcal{S})$. The stalk $\mathcal{R}_z$ is the zero module since $(s_1)_z,\dots,(s_\lambda)_z$ are *free* generators of $\mathcal{S}_z$; since $\mathcal{R}$ is itself coherent it follows that 0 generates the stalks $\mathcal{R}_\zeta$ for all ζ in some neighborhood V of z, $V\subset V_1$. So $\mathcal{R}=0$ in V, which says that $\mathcal{S}|V$ is isomorphic to $(\mathcal{O}_V)^\lambda$.

We now consider two cases. The first is that in which $\Phi^{-1}(z)$ consists of λ distinct points, say $w_1, \ldots, w_\lambda$. Then there is a neighborhood V of z and there are pairwise disjoint neighborhoods W_i of w_i such that Φ maps each W_i biholomorphically onto V. Let $\psi_i\colon V \to W_i$ be inverse to Φ. Every germ $s_z \in \mathcal{S}_z$ is represented by a λ-tuple of functions $(f_1, \ldots, f_\lambda)$, where each f_i is holomorphic in some neighborhood of w_i. The map $s_z \to (\tilde{f}_1, \ldots, \tilde{f}_\lambda)$, where $\tilde{f}_i$ is the germ at z of the function $f_i \circ \psi_i$, is an isomorphism of $\mathcal{S}_z$ with $(\mathcal{O}_z)^\lambda$.

The case in which $\Phi^{-1}(z)$ consists of fewer than λ points is not quite so easy. Fix one $w \in \Phi^{-1}(z)$ and let μ be the branching order of Φ at w [5, p. 103]. Let $H \subset \mathcal{O}_{\Omega',\mathrm{w}}$ consist of the germs at w of the functions $g \circ \Phi$, where g is holomorphic near z. Since H is isomorphic to $\mathcal{O}_{\Omega,\mathrm{z}}$, *we have to show that* $\mathcal{O}_{\Omega',\mathrm{w}}$ *is a free* module over H, of rank μ.

Let $\tilde{H}$ and $\tilde{\mathcal{O}}_{\Omega',\mathrm{w}}$ be the quotient fields of H and $\mathcal{O}_{\Omega',\mathrm{w}}$. We claim that

(i) $\mathcal{O}_{\Omega',\mathrm{w}}$ is the integral closure of H in $\tilde{\mathcal{O}}_{\Omega',\mathrm{w}}$, and

(ii) $\tilde{\mathcal{O}}_{\Omega',\mathrm{w}}$ is a finite algebraic extension of $\tilde{H}$.

Since $\mathcal{O}_{\Omega,\mathrm{z}}$ and $\mathcal{O}_{\Omega',\mathrm{w}}$ (hence also H) are unique factorization domains [5, p. 72] they are integrally closed [14, p. 261] and therefore (i) and (ii) will imply that $\mathcal{O}_{\Omega',\mathrm{w}}$ is a finite H-module [14, p. 265, Corollary 1](3). But since $\mathcal{O}_{\Omega,\mathrm{z}}$ and $\mathcal{O}_{\Omega',\mathrm{w}}$ are just the rings of convergent power series in N complex variables, it then follows from [4, Korollar 5] that $\mathcal{O}_{\Omega',\mathrm{w}}$ is actually free over H. The rank of $\mathcal{O}_{\Omega',\mathrm{w}}$ over H must then clearly be equal to the branching order of Φ at w, and this is the desired conclusion.

We turn to the proof of (i) and (ii). The point w has a neighborhood basis $\{V_\alpha\}$ such that $(V_\alpha, \Phi|V_\alpha, \Phi(V_\alpha))$ is an analytic cover of multiplicity μ. Hence [5, p. 104] every $f \in \mathcal{O}_{\Omega',\mathrm{w}}$ satisfies a monic polynomial equation

$$f^\mu + h_{\mu-1} f^{\mu-1} + \cdots + h_0 = 0 \qquad (h_i \in H). \tag{1}$$

Thus $\mathcal{O}_{\Omega',\mathrm{w}}$ is a subset of the integral closure of H in $\tilde{\mathcal{O}}_{\Omega',\mathrm{w}}$. On the other hand, every $x \in \tilde{\mathcal{O}}_{\Omega',\mathrm{w}}$ which is integral over H is also (trivially) integral over the larger ring $\mathcal{O}_{\Omega',\mathrm{w}}$, and since $\mathcal{O}_{\Omega',\mathrm{w}}$ is integrally closed (as noted above), it follows that $x \in \mathcal{O}_{\Omega',\mathrm{w}}$. Hence (i) is true.

Since (1) holds for every $f \in \mathcal{O}_{\Omega',\mathrm{w}}$, the usual proof of the fact that the algebraic numbers form a field shows that every $x \in \tilde{\mathcal{O}}_{\Omega',\mathrm{w}}$ satisfies an equation

$$a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0 = 0 \qquad (a_i \in H,\ a_m \neq 0).$$

Multiplying this by a_m^{m-1}, we see that $a_m x$ is integral over H. By (i), $a_m x \in \mathcal{O}_{\Omega',\mathrm{w}}$, and hence $f = a_m x$ satisfies an equation of the form (1). In other words, every $x \in \tilde{\mathcal{O}}_{\Omega',\mathrm{w}}$ is algebraic over H, of degree $\leq \mu$. Pick $x_0 \in \tilde{\mathcal{O}}$ so that its degree over H is maximal. If there were an $x_1 \in \tilde{\mathcal{O}}_{\Omega',\mathrm{w}}$, $x_1 \notin \tilde{\mathcal{O}}_{\Omega',\mathrm{w}}(x_0)$, then the dimension of the field $\tilde{H}(x_0, x_1)$ would be larger than that of $\tilde{H}(x_0)$ (as vector spaces over $\tilde{H}$). The

(3) As the referee has pointed out, the fact that $\mathcal{O}_{\Omega',\mathrm{w}}$ is a finite H-module follows also from a general theorem on analytic algebras, [11, Theorem 1, p. 10].

theorem of the primitive element [14, p. 84] implies that $H(x_0, x_1)=H(x_2)$ for some $x_2 \in \tilde{\mathcal{O}}_{\Omega',\mathbf{w}}$. But then x_2 has larger degree over $\tilde{H}$ than x_0, a contradiction. Consequently, $\tilde{\mathcal{O}}_{\Omega',\mathbf{w}}=\tilde{H}(x_0)$. This proves (ii) and completes the lemma.

I.3. Corollary. *If $\Phi \in \mathscr{M}(\Delta, U^N)$ for some $\Delta \in \mathscr{K}_N$, if Ω is a neighborhood of U^N such that Φ maps $\Omega'=\Phi^{-1}(\Omega)$ properly onto Ω, and if Φ has multiplicity λ, then the sheaf $\mathscr{S}$ is free of rank λ on a neighborhood of $\bar{U}^N$.*

Proof. This is simply the fact that a locally free sheaf on a neighborhood of a closed polydisc is actually free over some neighborhood of the polydisc. This follows from Cartan's lemma on holomorphic matrices and is to be found in [10, p. 86] where, however, it is formulated in terms of vector bundles. The relation between vector bundles and locally free sheaves is discussed in [5].

We can now prove the main result of this section.

I.4. Theorem. *If $\Delta \in \mathscr{K}_N$ and if $\Phi \in \mathscr{M}(\Delta, U^N)$ has multiplicity λ, then there exist functions $F_1, \dots, F_\lambda$, holomorphic in a neighborhood of $\bar{\Delta}$, such that every f holomorphic in Δ has a unique representation of the form*

$$f=\sum_{i=1}^{\lambda}(g_i \circ \Phi)F_i \tag{2}$$

where $g_1, \dots, g_\lambda$ are holomorphic in U^N.

Moreover, if $f \in H^\infty(\Delta)$ then each $g_i \in H^\infty(U^N)$. If $f \in A(\Delta)$, then each $g_i \in A(U^N)$.

Proof. By Corollary I.3, there is an open polydisc $\Omega \supset \bar{U}^N$ such that, setting $\Omega'=\Phi^{-1}(\Omega)$, the sheaf $\mathscr{S}$ of Lemma I.2 is isomorphic to $(\mathcal{O}_\Omega)^\lambda$. Hence there are sections $\tilde{F}_1, \dots, \tilde{F}_\lambda \in \Gamma(\Omega, \mathscr{S})$ with the following property: if V is open, $V \subset \Omega$, then *every $f \in \Gamma(V, \mathscr{S})$ is uniquely expressible as $\sum g_i\tilde{F}_i$, with $g_i \in \Gamma(V, \mathcal{O}_V)=\mathcal{O}(V)$.*

If we apply this to $V=U^N$, lift the statement to Δ by means of Φ, and let F_i be the element of $\mathcal{O}(\Omega')$ which corresponds to the section $\tilde{F}_i \in \Gamma(\Omega, \mathscr{S})$, we obtain (2).

If we apply the same statement to $V=\Omega$, we obtain an analogue (2′) of (2), with $f \in \mathcal{O}(\Omega')$, $g_i \in \mathcal{O}(\Omega)$. If $z \in \Omega$ is such that $\Phi^{-1}(\Phi(z))$ consists of λ distinct points $w_1, \dots, w_\lambda$, then (2′) gives

$$f(w_k)=\sum_{i=1}^{\lambda} g_i(\Phi(z))F_i(w_k) \qquad (k=1, \dots, \lambda). \tag{3′}$$

Proper choice of $f \in \mathcal{O}(\Omega')$ shows that the ordered λ-tuple $(f(w_1), \dots, f(w_\lambda))$ can be any point of C^λ. The matrix $(F_i(w_k))$ must therefore have rank λ whenever $w_1, \dots, w_\lambda$ are distinct.

This last condition holds for every z in a certain neighborhood of T^N, since Φ is a local homeomorphism at every point of $\Phi^{-1}(T^N)$. Thus there is a neighborhood Y of $\Phi^{-1}(T^N)$ in which $\det(F_i(w_k))$ is bounded away from zero.

Now suppose $f \in H^\infty(\Delta)$. The equations (3′) have analogues (3), with $g_i \in \mathcal{O}(U^N)$. For $z \in \Delta \cap Y$, (3) can be solved for $g_i(\Phi(z))$, by Cramer's rule. The form of the solution shows that each g_i is bounded in $\Phi(\Delta \cap Y)$. This latter set contains all $(\zeta_1, \ldots, \zeta_N)$ with $r < |\zeta_j| < 1$ for some r and all j. Hence $g_i \in H^\infty(U^N)$.

If, in addition, f is continuous on $\Delta \cup \Phi^{-1}(T^N)$, then the above application of Cramer's rule shows that each g_i extends to a function continuous on $\Phi(\Delta \cap Y) \cup T^N$. It follows that g_i is continuous on $U^N \cup T^N$. Hence $g_i \in A(U^N)$.

This completes the proof of the theorem. We note, incidentally, that the last paragraph of the proof shows that if g is holomorphic in Δ and continuous on $\Delta \cup \Phi^{-1}(T^N)$, then $g \in A(\Delta)$.

Every $\Phi \in \mathscr{M}(\Delta, U^N)$ is, by definition, a local homeomorphism at each point of $\Phi^{-1}(T^N)$. We now show by an example that Theorem I.4 can fail (even for $N=1$) if Φ fails to be a local homeomorphism at just one point of $\Phi^{-1}(T^N)$.

I.5. Example. For $z \in \mathbf{C}$, put $\Phi(z)=(z^2-1)^{-1}$, and regard Φ as a map of the Riemann sphere S into itself. The curve γ on which $|\Phi| = 1$ is shaped like an infinity sign, and it meets the imaginary axis in only one point, namely the origin, which is a double point of the curve. Let Δ be the component of $S \backslash \gamma$ which contains the point at infinity. Thus $\Delta = \Phi^{-1}(U)$, where U is the open unit disc in the plane, and Φ maps Δ onto U in a two-to-one fashion. The mapping Φ is a local homeomorphism at every point of the boundary γ of Δ, except zero. We shall show that $A(\Delta)$ is not a finitely generated module over $\Phi^*A(U)$.

Suppose, on the contrary, that $B_1, \ldots, B_M \in A(\Delta)$, and that every $F \in A(\Delta)$ has a (not necessarily unique) representation of the form

$$F(z) = \sum_{i=1}^{M} B_i(z) f_i((z^2-1)^{-1}) \qquad (z \in \Delta, f_i \in A(U)). \tag{4}$$

Let $L\colon A(U)^M \to A(\Delta)$ be given by $L(f_1, \ldots, f_M) = \sum B_j f_j \circ \Phi$; our hypothesis is that L is onto though not necessarily one-to-one.

If $G \in H^\infty(\Delta)$, there exists a bounded sequence $\{G_n\}$ in $A(\Delta)$ which converges uniformly on compacta in Δ to G. To see this, let $\psi\colon \Delta \to U$ be a conformal (one-to-one) mapping such that

$$\lim_{t \to 0^+} \psi(it) = 1 \quad \text{and} \quad \lim_{t \to 0^+} \psi(it) = -1.$$

Let $\phi\colon U \to \Delta$ be inverse to ψ. The mapping ψ extends continuously to $\overline{\Delta} \backslash \{0\}$, and ϕ extends continuously to $\overline{U}$. Since $G \circ \phi \in H^\infty(U)$, there is a bounded sequence $\{g_n\}$ in $A(U)$ which converges uniformly on compacta in U to $G \circ \phi$. Define $\tilde{g}_n$ by $\tilde{g}_n(z) = g_n(z)(1-z^2)^{1/n}$. The sequence $\{\tilde{g}_n\}$ is again a bounded sequence in $A(U)$ which converges to $G \circ \phi$ uniformly on compacta in U. Since the functions $\tilde{g}_n$ vanish at 1 and -1, $G_n = \tilde{g}_n \circ \psi$ is a well-defined element of $A(\Delta)$. Then $\{G_n\}$ is a bounded sequence in $A(\Delta)$ which converges uniformly on compacta in Δ to G.

Since the operator L is onto, the open mapping theorem, together with a simple normal families argument, shows that if $F \in H^\infty(\Delta)$, then F can be expressed in the form (4) with suitable $f_1, \ldots, f_M \in H^\infty(U)$. We will obtain our contradiction by applying this fact to a particular $f \in H^\infty(\Delta)$.

The function Φ is negative on the imaginary axis, and $\lim_{t\to 0}\Phi(it)=-1$. Let $\{\tau_n\}_{n=1}^{\infty}$ be a sequence in $(-1, 0)$ which decreases to -1. For each n, let $\Phi^{-1}(\tau_n)=\{\sigma_n^+, \sigma_n^-\}$, σ_n^+ above the real axis, σ_n^- below. If τ_n approaches -1 fast enough, there will exist an $F_0\in H^\infty(\Delta)$ such that $F_0(\sigma_n^+)=+1$, $F_0(\sigma_n^-)=-1$. (One way to obtain such an F_0 is to note that if τ_n tends to -1 fast enough, the set $\{\sigma_n^+\}_{n=1}^{\infty}\cup\{\sigma_n^-\}_{n=1}^{\infty}$ will be an interpolation set for $H^\infty(\Delta)$. In this connection, see [7].) By the last paragraph, the function F_0 can be written in the form (4), and we have, for all n,

$$1 = F_0(\sigma_n^+) = B_1(\sigma_n^+)f_1(\tau_n)+\cdots+B_M(\sigma_n^+)f_M(\tau_n) \tag{5}$$

and

$$-1 = F_0(\sigma_n^-) = B_1(\sigma_n^-)f_1(\tau_n)+\cdots+B_M(\sigma_n^-)f_M(\tau_n). \tag{6}$$

Since the functions f_j are bounded in U, there exists a subsequence of $\{\tau_n\}$, call it $\{\tau_n'\}$ with the property that for each j,

$$\lim_{n} f_j(\tau_n') = \alpha_j$$

exists. The functions B_j are continuous at 0, so if we take limits along the sequence $\{\tau_n'\}$ in (5) and (6), we are led to the contradiction that $-1=\sum B_j(0)\alpha_j=1$. Thus $A(\Delta)$ is not finitely generated over $\Phi^*A(U)$.

Let us observe that in this example $H^\infty(\Delta)$ *is* a free module of rank two over $\Phi^*H^\infty(U)$. Denote by q one of the branches of $(z^2-1)^{-1/2}$ holomorphic in Δ. The function q effects a conformal (one-to-one) mapping of Δ onto U. Each $F\in H^\infty(\Delta)$ is uniquely expressible in the form $F=f\circ q$, $f\in H^\infty(U)$. If $f(\zeta)=\sum_{k=0}^{\infty} b_k\zeta^k$, then the functions f_1 and f_2 defined by

$$f_1(\zeta) = \sum_{k=0}^{\infty} b_{2k+1}\zeta^k \quad\text{and}\quad f_2(\zeta) = \sum_{k=0}^{\infty} b_{2k}\zeta^k$$

are both in $H^\infty(U)$, and we have $f(\zeta)=\zeta f_1(\zeta^2)+f_2(\zeta^2)$. Consequently, $F=qf_1\circ\Phi+f_2\circ\Phi$. Since the f_1 and f_2 in this decomposition are uniquely determined by F, it follows that $\{1, q\}$ is a free basis for $H^\infty(\Delta)$ over $\Phi^*H^\infty(U)$. Of course, q is not continuous on $\bar{\Delta}$. Our previous argument shows that $H^\infty(\Delta)$ cannot be generated as a module over $\Phi^*H^\infty(U)$ by finitely many elements of $A(\Delta)$.

We now turn to some corollaries of Theorem I.4.

I.6. Corollary. (a) *If $f\in A(\Delta)$, $\Delta\in\mathscr{K}_N$, then f can be approximated uniformly on $\bar{\Delta}$ by functions holomorphic on a neighborhood of $\bar{\Delta}$.*

(b) *If $F\in H^\infty(\Delta)$, $\Delta\in\mathscr{K}_N$, there is a bounded sequence in $A(\Delta)$ which converges uniformly on compacta in Δ to F.*

Proof. Let $\Phi\in\mathscr{M}(\Delta, U^N)$ and let $G_1,\dots,G_\lambda\in\mathscr{O}(\bar{\Delta})$ constitute a free basis for $H^\infty(\Delta)$ over $\Phi^*H^\infty(U^N)$. If $F\in H^\infty(\Delta)$, write

$$F = \sum G_j\cdot(f_j\circ\Phi),\quad f_j\in H^\infty(U^N).$$

For $m=2, 3, \ldots$, let $f_j^{(m)}(z)=f_j((1-1/m)z)$ for $z\in U^N$, and set

$$F_m = \sum G_j\cdot(f_j^{(m)}\circ\Phi).$$

The sequence $\{F_m\}$ is a bounded sequence of functions holomorphic on $\bar{\Delta}$, and it converges uniformly on compacta in Δ to F. If $F\in A(\Delta)$, then the functions f_j lie in $A(U^N)$, and the sequence $\{F_m\}$ converges uniformly on $\bar{\Delta}$ to F.

It would be of interest to determine whether or not given $F\in H^\infty(\Delta)$, the sequence $\{F_m\}$ can be chosen to satisfy $\|F_m\|\leqq\|F\|$. This may be the case, but we have not proved it.

As noted in [13], theorems like Theorem I.4 can be used to prove certain extension theorems for functions in $A(\Delta)$ and $H^\infty(\Delta)$ when Δ is embedded in a polydisc. We have the following fact.

I.7. THEOREM. *Let $\Delta\in\mathscr{K}_N$, let Ω be a neighborhood of $\bar{\Delta}$ which is carried biholomorphically onto an analytic submanifold V of a neighborhood Ω' of $\bar{U}^M$ by the map Ψ, and let $\pi\colon U^M\to U^N$ be the projection which takes $(z_1,\ldots,z_M)$ to $(z_1,\ldots,z_N)$. If $\Delta=\Psi^{-1}(V\cap U^M)$, and if $\pi\circ\Psi\in\mathscr{M}(\Delta, U^M)$, then there exists a bounded linear operator $E\colon H^\infty(\Delta)\to H^\infty(U^M)$ which carries $A(\Delta)$ into $A(U^M)$ and which is an extension operator in the sense that $(Ef)\circ\Phi=f$ for all $f\in H^\infty(\Delta)$.*

Proof. Let $F_1,\ldots,F_\lambda\in\mathscr{O}(\bar{\Delta})$ be a free basis for $H^\infty(\Delta)$ over $(\pi\circ\Psi)^*H^\infty(U^N)$. Since the functions F_j are holomorphic on a neighborhood of $\bar{\Delta}$, and since V is an analytic submanifold of Ω', it follows [5, p. 245] that for some functions $G_1,\ldots,G_\lambda$ holomorphic on a neighborhood of $\bar{U}^M$, we have $F_j=G_j\circ\Psi$. We construct E as follows: If $f\in H^\infty(\Delta)$ and $f=\sum F_j(f_j\circ\pi\circ\Psi)$ with $f_j\in H^\infty(U^N)$, define $Ef\in H^\infty(U^M)$ by $Ef=\sum G_jf_j\circ\pi$. The operator E so defined is plainly linear and continuous. The choice of the functions G_j and f_j shows that $Ef\circ\Psi=f$.

If we set $\Delta'=\Psi(\Delta)$, this theorem shows that each bounded holomorphic function f on Δ' extends to a bounded holomorphic function F on U^M and, moreover, that if f has continuous boundary values, the extension F will also have continuous boundary values. As noted in [13, Example III.6] the norm of the operator will, in general, exceed one. (Unfortunately, this example is not correct as stated, for the functions ψ_δ are not one-to-one. We obtain a correct example if we redefine ψ_δ by means of

$$\psi_\delta(\zeta) = \left(\zeta^2, \zeta^3, \left(\frac{\zeta-\delta}{1-\delta\zeta}\right)^3\right).$$

Theorem II.9 of the present paper shows that no example with just two Blaschke products can exist.)

In general, $E(1)$ is not the function identically one on U^N, for the functions G_j can very well have common zeros in U^N.

In connection with this theorem, we should point out that in [3] Bishop has shown the existence of linear solutions to certain extension problems. See especially Theorem 7.II$_1$ and the note added in proof.

I.8. COROLLARY. *If $\Delta'=\Psi(\Delta)$, the ideal $I=\{f\in H^\infty(U^M) : f|\Delta'=0\}$ is a direct summand in $H^\infty(U^M)$, and $I\cap A(U^M)$ is a direct summand in $A(U^M)$.*

Proof. Given $f\in H^\infty(U^M)$, define Pf by

$$Pf = f - E(f\circ\Psi).$$

The operator P is a continuous projection in $H^\infty(U^M)$ whose range is the ideal I and which carries $A(U^M)$ onto $I\cap A(U^M)$. The existence of such projections implies the corollary.

II. **Embedding polydiscs in polydiscs.** In this section we will investigate the properties of mappings which embed polydiscs in higher dimensional polydiscs. We begin with a pair of lemmas.

II.1. LEMMA. *Suppose Ω is a connected open set in C^N, $\{g_j\}$ is a sequence in $\mathcal{O}(\Omega)$ each member of which is bounded by one in modulus, and* $\lim_{j\to\infty} g_j(z_0)=\alpha$ *for some $z_0\in\Omega$ and some α with $|\alpha|=1$. Then* $\lim g_j(z)=\alpha$ *uniformly on compacta in Ω.*

A standard normal families argument guarantees the conclusion for some subsequence of $\{g_j\}$; the lemma shows that it is unnecessary to pass to a subsequence.

Proof. If K is compact in Ω, there exists a compact set $H\subset U$, the unit disc in C, such that if $g\in\mathcal{O}(\Omega)$ vanishes at z_0 and is bounded by one on Ω, then $g(K)\subset H$. Set

$$\phi_w(z) = (z-w)/(1-\bar{w}z), \qquad \psi_w(z) = (z+w)/(1+\bar{w}z)$$

and $f_j(z)=\phi_{w_j}(g_j(z))$ where $w_j=g_j(z_0)$. Then $f_j(K)\subset H$, and since $g_j=\psi_{w_j}\circ f_j$, we see that $g_j(K)\subset\psi_{w_j}(H)$. Since $w_j\to\alpha$, $|\alpha|=1$, the sequence $\{\psi_{w_j}\}$ converges to α uniformly on compacta in U, and the result follows.

II.2. LEMMA. *If $\Phi: U^N\to U^M$ is a proper holomorphic map, then $N\leqq M$.*

Proof. If $M<N$, then since Φ is a closed mapping, a result from dimension theory [8, p. 91] provides a point $z\in U^M$ such that $\Phi^{-1}(z)$ is of positive dimension. Since Φ is proper and holomorphic, $\Phi^{-1}(z)$ is a compact subvariety of U^N. Since compact subvarieties of U^N are necessarily finite sets, we have a contradiction, and the lemma is proved.

The proper holomorphic maps of U into U are the finite Blaschke products. Our next theorem shows that proper holomorphic maps of a U^k into a U^n are also of a rather special form. If $f\in\mathcal{O}(U^k)$ and $z\in T^k$, we shall denote by $f^*(z)$ the limit $\lim_{r\to 1^-} f(rz)$ provided this limit exists. If f is bounded, $f^*(z)$ exists for almost all $z\in T^k$. (See, e.g., [15].)

II.3. THEOREM. *Let $\Phi=(\phi,\dots,\phi_n)$ be a proper holomorphic map of U^k into U^n. Then $k\leqq n$, and the functions $\phi_1,\dots,\phi_n$ can be so permuted that for $1\leqq j\leqq k$,*

(1) *ϕ_j depends only on z_j and is nonconstant, and*

(2) *$|\phi_j^*|=1$ on a set of positive measure in T, the unit circle. Moreover, if one of the following conditions* (a), (b) *or* (c) *is satisfied, then the functions $\phi_1,\dots,\phi_k$ are finite Blaschke products:*

(a) $k=n$.

(b) Φ *is continuous on* $\bar{U}^k$ *and* $\Phi(T^k)\subset T^n$.

(c) Φ *is holomorphic on a neighborhood of* $\bar{U}^k$.

Simple examples show that a proper holomorphic map $\Phi\colon U^k \to U^n$ need not satisfy any of the conditions (a), (b) or (c). For instance, let ϕ_1 be a conformal, one-to-one map of U onto the right half of U so that $\phi_1(1)=1$, $\phi_1(i)=i$, $\phi_1(-i)=-i$ and let ϕ_2 be a conformal map of U onto the left half of U so that $\phi_2(-1)=-1$, $\phi_2(-i)=-i$, and $\phi_2(i)=i$. Then $\Phi=(\phi_1,\phi_2)$ is a proper map of U to U^2 but neither ϕ_1 nor ϕ_2 is a finite Blaschke product.

Proof of the Theorem. We know from Lemma II.2 that $k\leqq n$.

If $z=(z_1,\dots,z_k)$, set $z'=(z_2,\dots,z_k)$, and put $0'=(0,\dots,0)\in C^{k-1}$. For each i and almost all $\zeta\in T$, the limit

$$c_i(\zeta) = \lim_{r\to 1^-} \phi_i(r\zeta, 0')$$

exists, and since Φ is proper, $|c_i(\zeta)|=1$ for at least one i. Let $E_i=\{\zeta\in T : |c_i(\zeta)|=1\}$. Then E_i has positive measure for at least one i since $E_1\cup\cdots\cup E_k$ covers almost all of T. By re-indexing if necessary, we may suppose E_1 to have positive measure, so that

$$\lim_{r\to 1^-} \phi_1(r\zeta, 0') = c_1(\zeta)$$

is of modulus one on a set of ζ's of positive measure. By Lemma II.1, applied to U^{k-1}, it follows that

$$\lim_{r\to 1^-} \phi_1(r\zeta, z') = c_1(\zeta) \tag{3}$$

for all $z'\in U^{k-1}$, $\zeta\in E_1$. If we define $g_{z'}(\lambda)=\phi_1(\lambda,z')$, then $g_{z'}\in H^\infty(U)$ for each $z'\in U^{k-1}$, and (3) implies that the radial limit of $g_{z'}-g_{w'}$ vanishes on E_1 whenever $z',w'\in U^{k-1}$. Since E_1 has positive measure, we have $g_{z'}=g_{w'}$, i.e., $\phi_1(\lambda,z')$ depends only on λ. If ϕ_1 were constant, it would have to be of modulus one which is impossible since Φ carries U^k into U^n. The variables $z_2,\dots,z_n$ can be dealt with in a similar way, so we have (1) and (2) of the theorem.

If $k=n$, it follows, after a permutation of indices, that

$$\Phi(z_1,\dots,z_k) = (\phi_1(z_1),\dots,\phi_k(z_k))$$

and in particular that $\Phi(z_1,0,\dots,0)=(\Phi_1(z_1),\phi_2(0),\dots,\phi_k(0))$. Since Φ is proper, ϕ_1 must be a proper holomorphic map of U to U, i.e., it must be a finite Blaschke product.

We now consider the case that Φ is holomorphic on a neighborhood of $\bar{U}^k$. The fact that in this case $\phi_1,\dots,\phi_k$ must be finite Blaschke products is an immediate consequence of the following result.

II.4. Lemma. *If f is holomorphic on a neighborhood of $\bar{U}^k$ and if $|f|=1$ on a set of positive measure in T^k, then $|f|$ is identically one on T^k.*

Proof. Define a map $Q: R^k \to T^k$ by

$$Q(t_1, \ldots, t_k) = (e^{it_1}, \ldots, e^{it_k}).$$

The function $1-|f \circ Q|^2 = 1-(f \circ Q)\overline{(f \circ Q)}$ is real analytic on R^k and vanishes on a set of positive measure. Consequently, it vanishes identically, and thus $|f|=1$ on T^k.

It remains only to consider the case (b). Under the hypotheses of (b), each ϕ_j is continuous on $\bar{U}^k$ and has modulus one on T^k. For $1 \leqq j \leqq k$, ϕ_j depends only on z_j and it follows that ϕ_j must, in fact, be a finite Blaschke product. This concludes the proof of the theorem.

It is interesting to observe that if Φ satisfies condition (b), it automatically satisfies condition (c) as our next lemma shows.

II.5. Lemma. *If $f \in A(U^k)$ has modulus one on T^k, then f is holomorphic on a neighborhood of $\bar{U}^k$.*

Proof. By Theorems 2.1 and 2.2 of [12], $f=P/Q$, P and Q relatively prime polynomials, Q free of zeros in U^k. Equation 2.1.2 of the same reference implies that Q is free of zeros on T^k. It follows that Q is free of zeros in $\bar{U}^k$, for otherwise $1/Q$ would violate the maximum modulus theorem.

Using Theorem II.3 and results from §I, we can establish the following fact.

II.6. Theorem. *Let $\Phi=(\phi_1, \ldots, \phi_N)$ be a holomorphic map of a neighborhood of $\bar{U}^k$ into C^N which is proper on U^k, nonsingular and one-to-one on $\bar{U}^k$ and which carries U^k into U^N. Then there is a continuous linear operator $E: H^\infty(U^k) \to H^\infty(U^N)$ which carries $A(U^k)$ into $A(U^N)$ and which satisfies $Ef \circ \Phi = f$.*

Proof. By Theorem II.3, we can reindex the functions ϕ_j so that

$$\phi_j(z) = B_j(z_j) \qquad i \leqq j \leqq k,$$

where each B_j is a nonconstant finite Blaschke product.

Let $\Psi: U^k \to U^k$ be given by $\Psi(z)=(B_1(z_1), \ldots, B_k(z_k))$. We assert that the Ψ so defined is in $\mathscr{M}(U^k, U^k)$. It surely is holomorphic on a neighborhood of $\bar{U}^k$. It is nonsingular at each point of T^k, for the Jacobian $\det(\partial\phi_j/\partial z_k)$ is simply $B_1'(z_1) \cdots B_k'(z_k)$, and since each of the derivatives B_j' is zero free on the unit circle, this Jacobian cannot vanish on T^k. If we let $D_R=\{(z_1, \ldots, z_k) : |B_j(z_j)| < R\}$, then for R larger than but sufficiently near one, Ψ will be holomorphic on D_R and will map D_R properly onto a neighborhood of $\bar{U}^k$. Thus $\Psi \in \mathscr{M}(U^k, U^k)$.

Since $\Psi \in \mathscr{M}(U^k, U^k)$, Theorem I.4 implies that $A(U^k)$ is a free module of rank λ, the multiplicity of Ψ, over $\Psi^*A(U^k)$. Let $\{F_1, \ldots, F_\lambda\}$ be a free basis for $A(U^k)$ over $\Psi^*A(U^k)$, each F_j holomorphic on a neighborhood of $\bar{U}^k$. Then $\{F_1, \ldots, F_\lambda\}$ is also a free basis for $H^\infty(U^k)$ over $\Psi^*H^\infty(U^k)$.

If we make R small enough, the neighborhood D_R of the next-to-last paragraph will be contained in the domain of definition of each of the functions F_j and also in that of all the ϕ_j. Since Ψ carries D_R properly onto a neighborhood Ω of $\bar{U}^k$, it follows that Φ will carry D_R properly into $\Omega\times C^{N-k}$.

Since Φ is nonsingular at each point of $\bar{U}^k$, it is nonsingular on D_R if R is small enough. Finally, if R is small enough, Φ will be one-to-one on D_R. If not there is a sequence $\{R_n\}$ which decreases to one, and for each n a pair of points z_n and z_n' in D_{R_n}, $z_n\neq z_n'$, such that $\Phi(z_n')=\Phi(z_n)$. By passing to subsequences, we may suppose $\{z_n\}$ and $\{z_n'\}$ to converge to z_0 and z_0' respectively. We shall have z_0, $z_0'\in\bar{U}^k$ and $\Phi(z_0)=\Phi(z_0')$. Since Φ is one-to-one on $\bar{U}^k$, we must have $z_0=z_0'$. However, since Φ is nonsingular at each point of $\bar{U}^k$, there is a neighborhood W of z_0 in C^k on which Φ is one-to-one. Since z_n and z_n' are eventually in W, we have a contradiction. Thus Φ must be one-to-one on D_R for R near one. Consequently if R is near enough to one, the set $\Phi(D_R)$ in $\Omega\times C^{N-k}$ will in fact be a submanifold, say M, and $\Phi\colon D_R\to M$ will be a biholomorphic map.

Thus for some choice of $G_1,\dots,G_\lambda\in\mathcal{O}(\Omega\times C^{N-k})$ we have $F_j=G_j\circ\Phi$. The operator E defined by $Ef=\sum G_j\tilde{f}_j$, if $f=\sum F_jf_j\circ\Psi$ and $\tilde{f}_j(z_1,\dots,z_n)=f_j(z_1,\dots,z_k)$ has the desired properties.

Thus, if we embed U^k in U^N as a submanifold and if the embedding satisfies certain regularity conditions at the boundary, bounded holomorphic functions on the embedded U^k extend to bounded holomorphic functions on U^N. It is natural to ask if the boundary regularity is necessary. The following example shows that some condition is necessary.

II.7. Example. We will construct a proper nonsingular one-to-one map Φ from U to U^2 such that for some $f\in H^\infty(U^2)$ there is no $F\in H^\infty(U)$ with $f=F\circ\Phi$.

Let S be the spiral

$$\{re^{i\theta} : r = 1-1/\theta,\ \pi \leqq \theta < \infty\}.$$

Let $\Omega=U\backslash S$, and let h be a conformal (1-1) mapping of U onto Ω. The map h can be chosen so that it is continuous on $\bar{U}\backslash\{1\}$. Let $0<r_1<r_2<\cdots$ be the points at which S meets $(0, 1)$. Fix n for the moment, and put $\alpha(\varepsilon)=h^{-1}(r_n+\varepsilon)$, $\beta(\varepsilon)=h^{-1}(r_n-\varepsilon)$ where ε is small and positive. As $\varepsilon\to 0$, $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ tend to distinct points of ∂U. Hence we can choose $\varepsilon_n>0$ so small that the following properties hold: If $\xi_n=r_n+\varepsilon_n$, $\eta_n=r_n-\varepsilon_n$, $h(\alpha_n)=\xi_n$ and $h(\beta_n)=\eta_n$, then $1-|\alpha_n|<n^{-2}$, $1-|\beta_n|<n^{-2}$, and

$$|g(\xi_n)-g(\eta_n)| \leqq (|\alpha_n-\beta_n|/n)\,\|g\|_U \tag{7}$$

for all $g\in H^\infty(U)$. The choice of h shows that α_n, $\beta_n\to 1$ as $n\to\infty$.

Let B be the Blaschke product whose zero set is $\{\alpha_n\}_{n=1}^\infty\cup\{\beta_n\}_{n=1}^\infty$, and define

$$\Phi(z) = (B(z), h(z)).$$

Since h is one-to-one, h' is zero free, so Φ is one-to-one and nonsingular. It is also proper, for as $z\to 1$, $|h(z)|\to 1$ and as $z\to e^{i\theta}\neq 1$, $|B(z)|\to 1$.

Suppose there exists $F\in H^\infty(U^2)$ such that $F(\Phi(z))=z$. Let $g(w)=F(0, w)$, $w\in U$. Since $\Phi(U)$ contains the points $(0, h(\alpha_n))=(0, \xi_n)$ and $(0, h(\beta_n))=(0, \eta_n)$ at which F is α_n and β_n respectively, the inequality (7) implies that $\|g\|_U \geqq n$ for all n, an impossibility since F is assumed bounded.

In connection with this example, it should be mentioned that the disc $\Phi(U)$ is not the zero set of any $F \in H^\infty(U^2)$. The choice of the spiral S shows that $r_n = 1-(2n\pi)^{-1}$, so $\sum (1-r_n)=\infty$ whence $\sum (1-|\xi_n|)=\infty$. If $f \in H^\infty(U^2)$ vanishes on $\Phi(U)$, then $F(0, \xi_n)=0$ for all n. Hence, setting $g(\lambda)=F(0,\lambda)$, $g(\xi_n)=0$ for all n. Since $g \in H^\infty(U)$ and since $\sum(1-|\xi_n|)=\infty$, it follows that g vanishes identically, and so F vanishes on the set $\{0\}\times U$. Thus $\Phi(U)$ is the zero set of no $F \in H^\infty(U^2)$.

II.8. Remark. There are serious restrictions on the way in which a U^k can be embedded in a U^N if the embedding is required to be holomorphic on a neighborhood of $\bar{U}^k$. For instance, suppose Φ is holomorphic on $\bar{U}^k$ and carries U^k properly into U^N. Then Corollary II.3 shows that Φ is of the form

$$\Phi(z) = (B_1(z_1), \ldots, B_k(z_k), \phi_{k+1}(z), \ldots, \phi_N(z))$$

where the B_j are finite Blaschke products. Consequently, if none of the B_j are one-to-one, and if $N \leqq 2k-1$, Φ cannot be nonsingular at every point of U^k.

Another result of this same nature is contained in the following theorem.

II.9. Theorem. *Suppose B_1 and B_2 are finite Blaschke products of multiplicities $1+k_1$ and $1+k_2$, $k_1>0$, $k_2>0$. Then either B_1' and B_2' have a common zero in U or else the pair (B_1, B_2) does not separate points on $\bar{U}$.*

Proof. Let $\alpha_1(z), \ldots, \alpha_{k_1}(z)$ be the points other than z for which $B_1(\alpha_i(z))=B_1(z)$. (If $B_1'(z)=0$, we allow some of the $\alpha_i(z)$ to be z, the number to depend on the multiplicity of B_1 at z.) Define $\beta_j(z)$ in a similar way: $B_2(\beta_j(z))=B_2(z)$. The function R defined by

$$R(z) = \prod_{i=1}^{k_1}\prod_{j=1}^{k_2} \{\alpha_i(z)-\beta_j(z)\}$$

is holomorphic in a neighborhood of $\bar{U}$. If $|z|=1$, then $|\alpha_i(z)|=|\beta_j(z)|=1$, so

$$\begin{aligned}\bar{R} &= \prod_{i,j}(\bar{\alpha}_i-\bar{\beta}_j) = \prod_{i,j}\left(\frac{1}{\alpha_i}-\frac{1}{\beta_j}\right)\\ &= \prod_{i,j}(\beta_j-\alpha_i)\left(\prod_i \alpha_i\right)^{-k_2}\left(\prod_j \beta_j\right)^{-k_1}\\ &= (-1)^{k_1k_2}R\left(\prod_i \alpha_i\right)^{-k_2}\left(\prod_j \beta_j\right)^{-k_1}.\end{aligned}$$

Assume for the moment that R has no zero on $|z|=1$. Then

$$R(z)(\overline{R(z)})^{-1} = (-1)^{k_1k_2}\left(\prod_{i=1}^{k_1}\alpha_i(z)\right)^{k_2}\left(\prod_{j=2}^{k_2}\beta_j(z)\right)^{k_1} \qquad (|z|=1).$$

As z traverses the unit circle once in the positive direction, the argument of the right side increases by $2\pi 2k_1k_2$, and the argument of the left side increases by

$2\Delta_{|z|=1} \arg R$. Consequently $\Delta_{|z|=1} \arg R = 2\pi k_1 k_2$ so that R has $k_1 k_2$ zeros in U. In any event, R has a zero in $\bar{U}$, say $R(z_0)=0$. Hence there are i and j such that $\alpha_i(z_0)=\beta_j(z_0)$; let this common value be w_0. If $w_0=z_0$, then $B_1'(z_0)=B_2'(z_0)=0$. Otherwise we have distinct points z_0, w_0 which are not separated by (B_1, B_2).

This theorem can be rephrased by saying that if two finite Blaschke products generate the Banach algebra $A(U)$ then one of them does by itself.

However, there exist three finite Blaschke products which generate $A(\bar{U})$ although no two of them do. See [13, Theorem IV.1].

III. Extending functions from embedded Reimann surfaces. In this short concluding section, we generalize Theorem II.1 of [13]. By a *finite open Riemann surface* we mean a Riemann surface R obtained by deleting from a compact surface R_0 a finite collection of disjoint closed discs with analytic bounding curves.

III.1. THEOREM. *Let R be a finite open Riemann surface with boundary $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_M$, each Γ_j an analytic simple closed curve. Let $\Phi: R \to U^N$ be a map which is holomorphic on some neighborhood V of $\bar{R}$, which is proper on R, and which embeds V as an analytic submanifold of a neighborhood of $\bar{U}^N$. Given $f \in A(R)$ $(H^\infty(R))$, there is $F \in A(U^N)$ $(H^\infty(U^N))$ such that $F \circ \Phi = f$.*

Let $\Phi = (\phi_1, \ldots, \phi_N)$. In [13], this theorem was proved under the additional assumption that one of the ϕ_j satisfies $|\phi_j| \equiv 1$ on Γ. Since such a ϕ_j lies in $\mathscr{M}(R, U)$, this case is also included in the results of our §I.

The proof of the theorem depends on a simple lemma.

III.2. LEMMA. *If $\Phi = (\phi_1, \ldots, \phi_N)$ is as in the statement of the theorem, then for each j there is a k such that $|\phi_k| \equiv 1$ on Γ_j.*

Proof. The mapping Φ is proper, so if $\zeta \in \Gamma_j$, there is $k(\zeta)=k$ such that $|\phi_k(\zeta)|=1$. Set $E_k = \{\zeta \in \Gamma_j : |\phi_k(\zeta)| = 1\}$. At least one E_k is uncountable; suppose E_1 is. Since Γ_j is an analytic simple closed curve, there is a real analytic map ψ from the real line onto Γ_j which is locally a homeomorphism. Then $1-|\phi_1 \circ \psi|^2$ is real analytic and has uncountably many zeros. Consequently it vanished identically, so $|\phi_1| \equiv 1$ on Γ_j.

Proof of the Theorem. Observe that if $\phi \in \mathcal{O}(\bar{R})$, $|\phi| \equiv 1$ on Γ_j and $|\phi| < 1$ on R, then $d\phi$ is zero free on Γ_j. (See [13, p. 367].)

Let R_0 be the compact surface from which we obtain R, and let $w(P, Q)$ be a Cauchy kernel for R_0 which is holomorphic on a neighborhood of $\bar{R}$. (See [9, Appendix].) Then if $f \in A(R)$, we can write

$$f(Q) = \frac{1}{2\pi i} \int_\Gamma f(P) w(P, Q) = \sum_{j=1}^{M} \frac{1}{2\pi i} \int_{\Gamma_j} f(P) w(P, Q).$$

Let f_j denote the jth summand. It lies in $A(R)$ and is, moreover, holomorphic on a neighborhood of Γ_k if $k \neq j$.

We will write $f_j = e_j + h_j$ where h_j is holomorphic on a neighborhood of $\bar{R}$ and where e_j is of the form $E_j \circ \Phi$ for some $E_j \in A(U^N)$. Since h_j is holomorphic on a neighborhood of $\bar{R}$, $h_j = H_j \circ \Phi$ for some $H_j \in \mathcal{O}(\bar{U}^N)$ so the proof of the decomposition $f_j = e_j + h_j$ is sufficient to establish the theorem.

Consider f_1. Let ψ_1 be one of the ϕ_k which is identically one in modulus on Γ_1. Thus ψ_1 maps Γ_1 in a μ-to-one manner onto the unit circle, and since $d\psi_1$ has no zeros on Γ_1, there is an annulus $B_1 \subset R$ one of whose bounding curves is Γ_1, and which is mapped in a μ-to-one fashion by ψ_1 onto an annulus

$$B = \{z \in \boldsymbol{C} : 1 > |z| > \varepsilon\}$$

for some $\varepsilon > 0$. By the theory of Alling [1], there exist functions $g_1, \dots, g_\mu$ holomorphic on a neighborhood of $\bar{B}_1$ which constitute a free basis for $A(B_1)$ over $\psi_1^* A(B)$. Let $f_1 = \sum_{m=1}^{\mu} g_m \tilde{f}_m^{(1)} \circ \psi_1$, $\tilde{f}_m^{(1)} \in A(B)$.

If ε is chosen close enough to one, the set $\psi_1^{-1}(B)$ will be a union $B_1 \cup S$ where $\bar{B}_1 \cap \bar{S} = \varnothing$. Then for a suitable neighborhood Ω of $\bar{B} \times U^{N-1}$, $\Phi(V) \cap \Omega$ will be a submanifold of Ω which decomposes into an annulus containing and only slightly larger than $\Phi(B_1)$, call this piece M_1, and another piece, M_2, which contains $\Phi(\bar{S})$. If Ω is small enough, $\Phi_1^{-1}(M_1)$ will lie in the domain of definition of all the g_m. Let $G_j \in \mathcal{O}(\Omega)$ be such that $G_j \circ \Phi = g_j$ on $\bar{B}_1$ and $G_j \circ \Phi = 0$ on S. (The set S may be empty; in this case, we may disregard the second condition imposed on G_j.)

Define F_1 on $\bar{B} \times \bar{U}^{N-1}$ by

$$F_1(z_1, \dots, z_N) = \sum G_m(z_1, \dots, z_N) \tilde{f}_m^{(1)}(z_1).$$

This function is in $A(B \times U^{N-1})$, and for fixed $z_1 \in \bar{B}$, it is holomorphic in $(z_2, \dots, z_N)$ in a neighborhood of $\bar{U}^{N-1}$. We have the decomposition $F_1 = F_1^+ - F_1^-$ where

$$F_1^+(z_1, \dots, z_N) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{F_1(\zeta, z_2, \dots, z_N)}{\zeta - z_1} d\zeta$$

and

$$F_1^-(z_1, \dots, z_N) = \frac{1}{2\pi i} \int_{|\zeta|=\varepsilon} \frac{F_1(\zeta, z_2, \dots, z_N)}{\zeta - z_1} d\zeta.$$

On B_1, we have $f_1 = F_1^+ \circ \Phi - F_1^- \circ \Phi$. Since $F_1^+ \in A(U^N)$, it follows that $F_1^- \circ \Phi$ continues to an element h_1 of $A(R)$. We assert that h_1 is holomorphic on a neighborhood of $\bar{R}$. Consider $\zeta_0 \in \partial R$. Two cases are possible. It may be that $|\psi_1(\zeta_0)| < 1$. Since for each ζ with $|\zeta| = 1$, $F_1(\zeta, z_2, \dots, z_N)$ is holomorphic on a neighborhood of $\bar{U}^{N-1}$, the formula for F_1^+ shows it to be holomorphic in a neighborhood of $\Phi(\zeta_0)$. Since f_1 is holomorphic at ζ_0, it follows that h_1 is necessarily holomorphic there. If $|\psi_1(\zeta_0)| = 1$ and $\zeta_0 \in \bar{B}_1$, then $h_1(\zeta) = F_1^-(\Phi(\zeta))$ for ζ near ζ_0, and this is evidently holomorphic near ζ_0. If $|\psi_1(\zeta_0)| = 1$ and $\zeta_0 \notin \bar{B}_1$, then f_1 is holomorphic near ζ_0 and we have $h_1 = f_1 + F_1^+ \circ \Phi$. We have that $F_1 \circ \Phi \equiv 0$ near ζ_0, so it is enough to prove that near ζ_0, $F_1^- \circ \Phi$ is holomorphic. Again, since for ζ with $|\zeta| = \varepsilon$,

$F(\zeta, z_2, \ldots, a_N)$ is holomorphic on $\bar{U}^{N-1}$, this is immediate from the formula for F_1^-. This concludes the proof of the theorem.

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